Assignment 7 supplement

<u>Proof rules for the Universal Quantifier</u>

Universal Elimination - ∀E

This rule represents the inference from a universally quantified claim to any instance of it. For example, from ∀xPx we can infer by Pa, Pb, Pj, Pn, etc. The conclusion of the use of ∀E depends on the same assumptions as the premise. Keep the following points in mind when using ∀E:

1) In a correct use of ∀E, the universal quantifier must be the main connective. Thus it is not correct to apply ∀E to sentences like ~∀xPx or ∀xPx→Sa.

2) When applying ∀E to sentences where there is more than one occurrence of the variable quantified over, we must replace all occurrences of the variable and they must all be replaced by the same name. For example, from ∀xRxx we can infer Raa, Rbb, etc. but not Rxa, Rax, Rab, Rbc, etc.

3) If a sentence contains several initial quantifiers such as ∀x∀yRxy we may use two applications of ∀E to replace first the x and then the y. Here it is perfectly okay to use the same name twice if you like. So in two steps we could infer Raa, Rab, Rcc, etc.

Universal Introduction - ∀I

This rule allows the inference of a universally quantified claim from an arbitrary instance of that claim. Thus the rule allows the inference from Pa to ∀xPx as long as Pa is an arbitrary instance. To be arbitrary, it must be the case that the letter 'a' does not occur in any of the assumptions that Pa depends on. Just as with ∀E, the result of using ∀I depends on the same assumptions that its premise depended on. Keep the following points in mind when using ∀I:

1) In a correct use of ∀I the conclusion must have the universal quantifier as the main connective. For example, it is impossible to get ~∀xFx by using ∀I. If we had ~Fa as an arbitrary instance, we could use ∀I to infer ∀x~Fx.

2) If there is more than one occurrence of an arbitrary name in the premise of a use of ∀I, then the rule requires that we generalize on all occurrences of that name. So if we have Raa and 'a' is arbitrary, we can infer ∀xRxx but we could not infer ∀xRax or ∀xRxa.

3) Obviously it is a critical feature of a correct use of ∀I that the sentence we are generalizing over is an arbitrary instance. Before applying the ∀I rule, always make sure that the name you are generalizing over does not occur in any of the assumptions that the

line depends on. For example, to go from Rab to ∀xRxb we need to make sure that 'a' does not occur in any of the assumptions to the left of the line. If they contain 'b' that is perfectly okay since we are not generalizing on 'b.' If we wanted to prove ∀xRax we would need to make sure that 'b' was arbitrary.

<u>Strategy and Examples</u>

To prove a universally quantified sentence, try to prove an arbitrary instance of that sentence. To assure that the instance is arbitrary, choose a name that does not occur previously in the proof.

EXAMPLE 1 ∀x(Px → Qx), ∀x~Qx ├ ∀x~Px

Step 1. The conclusion is a universal claim. I will try to prove an arbitrary instance of it so that I can use ∀I. Since 'a' is nowhere in my proof yet, I will try to prove ~Pa.

1	(1)	∀x(Px → Qx)	A
2	(2)	∀x~Qx	A
		~Pa	new goal
		∀x~Px	∀I

Step 2. We can now use ∀E and MT to finish our proof. Line 6 is a correct use of ∀I because neither of the assumptions to the left of line 6 contain the name 'a'.

1	(1)	∀x(Px → Qx)	A
2	(2)	∀x~Qx	A
1	(3)	Pa→Qa	1 ∀E
2	(4)	~Qa	2 ∀E
1,2	(5)	~Pa	3,4 MT
1,2	(6)	∀x~Px	5 ∀I

EXAMPLE 2 ~∀x~(Ax&Bx) ├ ~∀x~Ax

Step 1. Since no other strategy is apparent I will assume the opposite of my goal and try to derive a contradiction in order to use RAA.

1	(1)	~∀x~(Ax&Bx)	A
2	(2)	∀x~Ax	A
		contradiction	
		~∀x~Ax	RAA

Step 2. What contradiction should I try to prove? I could try to contradict line 1. Since that is the negation of a complicated sentence, there is no other obvious way to use it. Now I set my new goal as ∀x~(Ax&Bx) which I will try to prove by using ∀I. To do this, I need to prove an arbitrary instance of it.

1	(1)	~∀x~(Ax&Bx)	A
2	(2)	∀x~Ax	A
		~(Aa & Ba)	new goal
		∀x~(Ax&Bx)	∀I
		~∀x~Ax	RAA

Step 3. To prove $\sim(Aa \,\&\, Ba)$, notice that by DeM this is equivalent to $\sim Aa \vee \sim Ba$. This follows from $\sim Aa$ which I can easily get by using line 2. Line 6 is a correct use of $\forall$I since the name ‘a’ does not occur in line 2.

1	(1) $\sim\forall x\sim(Ax\&Bx)$	A
2	(2) $\forall x\sim Ax$	A
2	(3) $\sim Aa$	2 $\forall$E
2	(4) $\sim Aa \vee \sim Ba$	3 vI
2	(5) $\sim(Aa \,\&\, Ba)$	4 DeM
2	(6) $\forall x\sim(Ax\&Bx)$	5 $\forall$I
1	(7) $\sim\forall x\sim Ax$	1,6 RAA (2)

EXAMPLE 3 $\forall x(Cx\to\forall y(Dy\to Gxy)) \vdash \forall x\forall y((Cx\&Dy)\to Gxy)$

Step 1. To prove a universal claim, prove an arbitrary instance of it. Lets use ‘a’ to replace ‘x’ in that instance. This is also a universal claim so I will try to prove an arbitrary instance of it. Here, I cannot the ‘y’ by ‘a’ so I will choose a different name. Let’s use ‘b’. Then we will end our proof with two uses of $\forall$I.

1	(1) $\forall x(Cx\to\forall y(Dy\to Gxy))$	A
	$(Ca\&Db)\to Gab$	new goal
	$\forall y((Ca\&Dy)\to Gay)$	$\forall$I
	$\forall x\forall y((Cx\&Dy)\to Gxy)$	$\forall$I

Step 2. Since our goal is now a conditional I will assume its antecedent and try to prove its consequent. We can prove the consequent by appropriate uses of $\forall$E together with our SL rules.

Line 10 is a correct use of $\forall$I since assumption 1 doesn’t contain the letter ‘b’ and line 11 is correct since 1 doesn’t contain ‘a’.

1	(1) $\forall x(Cx\to\forall y(Dy\to Gxy))$	A
2	(2) $Ca\&Db$	A
2	(3) Ca	2 &E
2	(4) Db	2 &E
1	(5) $Ca\to\forall y(Dy\to Gay)$	1 $\forall$E
1,2	(6) $\forall y(Dy\to Gay)$	3,5$\to$E
1,2	(7) $Db\to Gab$	6 $\forall$E
1,2	(8) Gab	4,7$\to$E
1	(9) $(Ca\&Db)\to Gab$	8$\to$I (2)
1	(10) $\forall y((Ca\&Dy)\to Gay)$	9 $\forall$I
1	(11) $\forall x\forall y((Cx\&Dy)\to Gxy)$	10 $\forall$I